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Flat portions on the boundary of the indefinite numerical range of 3 × 3 matrices

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Abstract

We focus on complex 3×3 matrices whose indefinite numerical ranges have a flat portion on the boundary. The results here obtained are parallel to those of Keeler, Rodman and Spitkovsky for the classical numerical range.

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1. Introduction

For $J = I_r \oplus -I_{n-r}$ $(0 < r < n)$, where I_m denotes the identity matrix of order m, consider $\mathbb{C}^n$ endowed with the Krein structure defined by the indefinite inner product $\langle \xi_1, \xi_2 \rangle_J = \xi_2^* J \xi_1,\ \ \xi_1, \xi_2 \in \mathbb{C}^n$. Let M_n be the algebra of $n \times n$ complex matrices. The J*-numerical range* of $A \in M_n$ is defined as

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$$W_J(A) = \left\{ \frac{\xi^* J A \xi}{\xi^* J \xi} : \xi \in \mathbb{C}^n, \xi^* J \xi \neq 0 \right\}.$$

If $J = \pm I_n$, then $W_J(A)$ reduces to the well-known *classical numerical range* of A, usually denoted by $W(A)$.

For $A \in M_n$, $W(A)$ is a compact and convex set [5], but $W_J(A)$ may not be closed and is either unbounded or a singleton [8,9,10,12]. On the other hand, $W_J(A)$ is the union of the convex sets

$$W_J(A) = W_J^+(A) \cup W_{-J}^+(A),$$

where

$$W_J^{\pm}(A) = \left\{ \xi^* J A \xi : \xi \in \mathbb{C}^n, \xi^* J \xi = \pm 1 \right\}$$

and $W_{-J}^+(A) = -W_J^-(A)$ [10,12].

For $A \in M_n$, we have $W_J(\alpha I_n + \beta A) = \alpha + \beta W_J(A)$, $\alpha, \beta \in \mathbb{C}$. A matrix A can be uniquely expressed as $A = H^J + \mathrm{i} K^J$, where $H^J = (A + JA^*J)/2$ and $K^J = (A - JA^*J)/(2i)$ are J-*Hermitian* matrices, that is, $H^J = J(H^J)^*J$ and $K^J = J(K^J)^*J$. Denoting by Re S and Im S the projection of $S \subseteq \mathbb{C}$ on the real and imaginary axes, respectively, we have Re $W_J(A) = W_J(H^J)$ and Im $W_J(A) = W_J(K^J)$.

The *supporting lines* of $W_J(A)$ are the supporting lines of the convex sets $W_J^+(A)$ and $W_{-J}^+(A)$. In [1,12], it was proved that if $ux + vy + w = 0$ is the equation of a supporting line of $W_J^+(A)$ ($W_{-J}^+(A)$), then the polynomial of Kippenhahn, $F_A^J(u, v, w) = \det(uH^J + vK^J + wI_n)$, satisfies

$$F_A^J(u, v, w) = 0. \tag{1}$$

Eq. (1), with u, v, w viewed as homogeneous line coordinates, defines an algebraic curve of class n on the complex projective plane $P_2(\mathbb{C})$ and its n real foci are the eigenvalues of A [3]. The real affine part of this curve is denoted by $C_J(A)$ and called the *associated curve* of $W_J(A)$. If $J = \pm I_n$, $C_J(A)$ is simply denoted by $C(A)$ and generates $W(A)$ as its convex hull [7]. The relation between $C_J(A)$ and $W_J(A)$ is described in [2,3]. For the degenerate cases, $W_J(A)$ may be a singleton, a line, a subset of a line, the whole complex plane, or the complex plane except a line. For the nondegenerate cases, $W_J(A)$ is the pseudo-convex hull of $C_J(A)$ defined as follows. Let $X = X^+ \cup X^-$ be a nonempty subset of $\mathbb{C}$, such that $X^+ \subseteq W_J^+(A)$ and $X^- \subseteq W_{-J}^+(A)$. For any pair of points p, q in X^+, or in X^-, take the closed line segment $[p, q]$, and for any pair of points p, q produced by vectors with J-norms of opposite sign take the two half-rays of the line defined by them with endpoints p, q. The set so obtained is called the *pseudo-convex hull* of X, denoted by PC$[X]$.

A matrix A is *essentially* J-*Hermitian* if there exist $\alpha, \beta \in \mathbb{C}$ such that $\alpha A + \beta I_n$ is J-*Hermitian*. Obviously, a matrix A is essentially J-Hermitian if and only $W_J(A)$ is a subset of a line. Let A be a non-essentially J-Hermitian matrix. Suppose that the straight line

$$\ell = \{(x, y) \in \mathbb{R}^2 : ax + by + c = 0, a, b, c \in \mathbb{R}\}$$

is a supporting line of $W_J(A)$. Let $\partial W_J(A)$ denote the boundary of $W_J(A)$. If $\ell \cap \partial W_J(A)$ contains more than one point, $\ell \cap \partial W_J(A)$ is called a *flat portion* on the boundary of $W_J(A)$. The definition of flat portions on $\partial W_J^+(A)$ (or on $\partial W_{-J}^+(A)$) is analogous. A matrix $U \in M_n$ is J-*unitary* if $U^{-1} = JU^*J$ and all $n \times n$ J-unitary matrices form a group denoted by $\mathscr{U}_{r,n-r}$. For any $U \in \mathscr{U}_{r,n-r}$, we have $W_J(A) = W_J(U^{-1}AU)$. We say that a matrix A is J-*unitarily reducible* if there exists a J-unitary matrix $U \in \mathscr{U}_{r,n-r}$ such that $U^{-1}AU = A_1 \oplus A_2$, $U^{-1}JU = J_1 \oplus J_2$, where $A_1, J_1 \in M_m$, $m \neq 0, n$, and we have $W_J(A) = \mathrm{PC}[W_{J_1}(A_1) \cup W_{J_2}(A_2)]$.

For a J-unitarily reducible matrix, the existence of flat portions on the boundary of its J-numerical range is a common occurrence. If A is J-normal with anisotropic eigenvectors, that is, eigenvectors ξ such that $\xi^* J\xi \neq 0$, then $W_J(A)$ is the pseudo-convex hull of the eigenvalues of A [2] and flat portions appear on $\partial W_J(A)$. The smallest value of n for which there exist J-unitarily irreducible matrices whose numerical ranges have a flat portion on $\partial W_J(A)$ is $n = 3$, and henceforth we concentrate on this case.

For $A \in M_2$, the elliptical range theorem [11] states that $W(A)$ is an elliptical disc (possibly degenerate) whose foci are the eigenvalues α_1 and α_2 of A, being the major and minor axis of length

$$\sqrt{\operatorname{Tr}(A^*A) - 2\operatorname{Re}(\overline{\alpha_1}\alpha_2)} \quad \text{and} \quad \sqrt{\operatorname{Tr}(A^*A) - |\alpha_1|^2 - |\alpha_2|^2},$$

respectively. In the indefinite case, for $A \in M_2$ and $J = \operatorname{diag}(1, -1)$, the hyperbolical range theorem [1] states that $W_J(A)$ is bounded by the hyperbola (possibly degenerate) with foci at α_1 and α_2, and transverse and non-transverse axis of length

$$\sqrt{\operatorname{Tr}(JA^*JA) - 2\operatorname{Re}(\overline{\alpha_1}\alpha_2)} \quad \text{and} \quad \sqrt{|\alpha_1|^2 + |\alpha_2|^2 - \operatorname{Tr}(JA^*JA)},$$

respectively.

The description of $W_J(A)$, when $A \in M_n$ and $n > 2$, is in general difficult. In certain cases, $W_J(A)$ is still an hyperbola and its interior, independently of the size of A. The 3×3 case was studied in [3] using the classification of $C_J(A)$ based on the factorability of $F_A^J(u, v, w)$. However, a constructive procedure allowing us to determine the shape of $W_J(A)$ for an arbitrary matrix $A \in M_3$ is not provided. In Section 2, we investigate J-unitarily irreducible matrices in M_3 having a flat portion on the boundary of the J-numerical range. In Section 3, we determine $W_J(A)$ for upper triangular matrices $A \in M_3$. The particularly simple case of triangular matrices with one-point spectrum is discussed. The results obtained here are inspired by those obtained by Keeler et al. for the classical numerical range [6].

2. *J*-unitarily irreducible 3 × 3 matrices with a flat portion on $\partial W_J(A)$

A flat portion on the boundary of the J-numerical range may be a (closed) line segment, two (closed) half-lines of the same line, a (closed) half-line or a whole line. The proof of the next result uses well-known formulas for the maximum number of singularities of an algebraic curve of order n (see, for example, [4, p. 49]).

Proposition 1. *For $A \in M_n$, with $n > 2$, the number of flat portions $l_J(A)$ on $\partial W_J(A)$ is less than or equal to $n(n-1)/2$. If $F_A^J(u, v, w)$ is irreducible, then*

$$l_J(A) \leqslant \frac{(n-1)(n-2)}{2}.$$

Proof. Each line originating a flat portion on $\partial W_J(A)$, $A \in M_n$, is a flexional tangent or a multiple tangent of $C_J(A)$. By dual considerations, we obtain a singular point of the dual curve of $C_J(A)$. Since $C_J(A)$ is a curve of class n, its dual curve has order n and the number of its singularities is less than or equal to $n(n-1)/2$. For an irreducible curve of order n, the upper bound is $(n-1)(n-2)/2$. $\square$

Proposition 2. *Let $A = H^J + \mathrm{i}K^J \in M_n$. If $\partial W_J(A)$ contains a flat portion, then for a certain real direction (u, v), $u = \cos\theta, v = \sin\theta, \theta \in \mathbb{R}$, the matrix $uH^J + vK^J$ has a multiple eigenvalue.*

Proof. By a translation and a rotation, we consider the flat portion on the imaginary axis. The imaginary axis defines a flat portion on $\partial W_J(A)$ if and only if it is a flexional tangent of $C_J(A)$ or a multiple tangent of the associated curve (at least) at two distinct points (the points can be finite or infinite, real or complex). Consider the dual curve of $C_J(A)$, defined in homogeneous point coordinates by

$$F_A^J(x, y, t) = \det(xH^J + yK^J + tI_n) = 0.$$

By dual considerations, if $x = 0$ is a flexional or a multiple tangent of $C_J(A)$, then (1:0:0) is a singular point of the dual curve, with multiplicity $m \geqslant 2$. It follows that

$$F_A^J(1, 0, 0) = \frac{\partial F_A^J}{\partial t}(1, 0, 0) = \cdots = \frac{\partial^{m-1} F_A^J}{\partial t^{m-1}}(1, 0, 0) = 0,$$

which implies that the coefficients $x^n, x^{n-1}t, \ldots, x^{n-(m-1)}t^{m-1}$ of the polynomial $F_A^J(x, y, t)$ vanish. Analyzing the solutions of the secular equation $\det(H^J - \lambda I_n) = 0$, we conclude that 0 is an eigenvalue of H^J with multiplicity at least m. $\square$

Throughout this section we assume that $J = \mathrm{diag}(1, 1, -1)$, and that $A \in M_3$ is a J-unitarily irreducible matrix written as $A = H^J + \mathrm{i}K^J$, where H^J and K^J are J-Hermitian matrices. To avoid trivial cases we also assume that A is not essentially J-Hermitian.

Theorem 1. *Let $J = \mathrm{diag}(1, 1, -1)$ and let $A \in M_3$ be a J-unitarily irreducible matrix. If $W_J(A)$ has a line segment on its boundary, then it lies on $\partial W_J^+(A)$. Analogously, if there exists a single half-line on $\partial W_J(A)$, then it lies on $\partial W_J^+(A)$.*

Proof. We prove (by contradiction) that the line segment on $\partial W_J(A)$ necessarily belongs to $\partial W_J^+(A)$. Indeed, assume that $W_{-J}^+(A)$ contains this line segment. After translation, rotation, and scaling of A, we may assume that the line segment has endpoints 0 and i. By Proposition 2, 0 is an eigenvalue of H^J with multiplicity at least 2. There exists $e_3 \in \mathbb{C}^n$ such that $e_3^* J e_3 = -1$ and $H^J e_3 = 0$. Consider also two vectors $e_1, e_2 \in \mathbb{C}^n$, $e_1^* J e_1 = e_2^* J e_2 = 1$, such that $\{e_1, e_2, e_3\}$ is a J-orthogonal basis of $\mathbb{C}^3$. The matrix representation of JH^J in this basis is

$$\begin{bmatrix} a & c & 0 \\ \bar{c} & b & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad a, b \in \mathbb{R}, \quad c \in \mathbb{C},$$

where $ab = |c|^2 \neq 0$, because A is not essentially J-Hermitian. Hence, under a J-unitary similarity transformation JH^J may be written as

$$JH^J = \begin{bmatrix} a' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $a' = a + b$. The quadratic form $\xi^* JH^J \xi$ vanishes if and only if $\xi = (0, \zeta, \eta) \in \mathbb{C}^3$. Let S be the subspace generated by e_2, e_3, and denote by $A' \in M_2$ the restriction of A to S. For $J' = \mathrm{diag}(1, -1)$, $W_{J'}(A')$ may be the real line, the real line except a point, or two half-rays.

Henceforth, it may not degenerate either to a half-line or to a line segment. Hence, $[0, \mathrm{i}]$ is contained in the boundary of $W_J^+(A)$.

To prove the second part of the theorem, we may suppose that the flat portion on $\partial W_J(A)$ is contained on the positive imaginary axis, and analogous arguments hold. □

Next, we derive a canonical form for an irreducible matrix with a closed line segment on the boundary of the J-numerical range.

Theorem 2. *Let $J = \operatorname{diag}(1, 1, -1)$ and let $A \in M_3$ be J-unitarily irreducible. Under J-unitary similarity, translation, rotation, and scaling, A may be written in the form*

$$A = \begin{bmatrix} \mathrm{i} & 0 & c_1 \\ 0 & 0 & c_2 \\ c_1 & c_2 & \psi \end{bmatrix}, \tag{2}$$

where c_1, c_2 are positive real numbers and $\operatorname{Re}\psi < 0$, if and only if $W_J(A)$ has a closed line segment on its boundary. In this form, $W_J^+(A)$ has the line segment $[0, \mathrm{i}]$ as a flat portion and is contained in the closed right half-plane.

Proof. ($\Rightarrow$) Assume that under J-unitary similarity, translation, rotation, and scaling, A is written in the form (2). Consider the Hermitian matrices

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\operatorname{Re}\psi \end{bmatrix} \quad \text{and} \quad JK^J = \begin{bmatrix} 1 & 0 & -\mathrm{i}c_1 \\ 0 & 0 & -\mathrm{i}c_2 \\ \mathrm{i}c_1 & \mathrm{i}c_2 & -\operatorname{Im}\psi \end{bmatrix}.$$

Since $\operatorname{Re}\psi < 0$, we have $W_{-J}^+(H^J) =]-\infty, \operatorname{Re}\psi]$, $W_J^+(H^J) = [0, +\infty[$, and so $W_J^+(A)$ is entirely contained in the right half-plane. Furthermore, $\xi^* JH^J \xi$ vanishes if $\xi = (\zeta, \eta, 0) \in \mathbb{C}^3$ and we get

$$\frac{\xi^* JK^J \xi}{\xi^* J \xi} = \frac{|\zeta|^2}{|\zeta|^2 + |\eta|^2}.$$

Thus, the interval $[0, 1]$ is described, and so the line segment $[0, \mathrm{i}]$ is contained in $W_J^+(A)$, being the imaginary axis a supporting line of $W_J^+(A)$.

($\Leftarrow$) Let $W_J^+(A)$ have a closed line segment as a flat portion on its boundary. After translation, rotation and scaling, we may assume that this line segment is $[0, \mathrm{i}]$. By Proposition 2, 0 is an eigenvalue of H^J with multiplicity at least 2. There exists $e_1 \in \mathbb{C}^n$ such that $e_1^* J e_1 = 1$ and $H^J e_1 = 0$. Consider two vectors $e_2, e_3 \in \mathbb{C}^n$, $e_2^* J e_2 = 1$, $e_3^* J e_3 = -1$, such that $\{e_1, e_2, e_3\}$ is a J-orthogonal basis of $\mathbb{C}^3$. In this basis, the matrix representation of JH^J is

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & c \\ 0 & \bar{c} & b \end{bmatrix}, \tag{3}$$

where a, b are real and c is a complex number satisfying $ab = |c|^2$. Since A is not an essentially J-Hermitian matrix, it is clear that $JH^J \neq 0$, and so $|c| \neq 0$. We prove (by contradiction) that $|a| \neq |c|$. Let $|a| = |c|$ and without loss of generality we may suppose $c > 0$. Two possibilities may occur: $a = b = c$ or $a = b = -c$. Assume that $a = b = c$. Since we have $\xi^* JH^J \xi = 0$ if $\xi = (1, \eta, -\eta) \in \mathbb{C}^3$, consider the matrix representation of JK^J in the basis $\{e_1, e_2, e_3\}$

$$JK^J = \begin{bmatrix} \alpha & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ \mathrm{i}\overline{\nu_2} & \mathrm{i}\overline{\nu_3} & \gamma \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \nu_1, \nu_2, \nu_3 \in \mathbb{C}$$

and the function

$$f(\xi) := \xi^* JK^J \xi = \alpha + (\beta + \gamma - 2\,\mathrm{Im}\,\nu_3)|\eta|^2 + 2|\eta||\nu_1 - \nu_2|\sin\phi,$$

where $\phi = \arg\eta + \arg(\nu_1 - \nu_2)$. This function reduces to a point if $\beta + \gamma - 2\,\mathrm{Im}\,\nu_3 = 0$ and $\nu_1 - \nu_2 = 0$, describes the whole real line if $\beta + \gamma - 2\,\mathrm{Im}\,\nu_3 = 0$ and $\nu_1 - \nu_2 \neq 0$, and a half-line of the real line if $\beta + \gamma - 2\,\mathrm{Im}\,\nu_3 \neq 0$. However, a line segment is never produced, contradicting the hypothesis. Then $|a| \neq |c|$, and so in a certain basis the matrix (3) is either of the form

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a' & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4}$$

or of the form

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a' \end{bmatrix} \tag{5}$$

with $a' = a - b$. It can be easily seen that the form (4) leads to a contradiction, because it is incompatible with the existence of a line segment on the boundary. Hence, we necessarily have (5). Thus, $W_J^+(H^J) = [0, +\infty[$ and $W_{-J}^+(H^J) =]-\infty, -a']$, being $-a' < 0$ since $W_J^+(A)$ is contained in the closed right half-plane.

The quadratic form $\xi^* JH^J \xi$ vanishes for $\xi = (\zeta, \eta, 0) \in \mathbb{C}^3$. Let A' be the principal submatrix of

$$A = H^J + \mathrm{i}K^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a' \end{bmatrix} + \mathrm{i} \begin{bmatrix} \alpha & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ -\mathrm{i}\overline{\nu_2} & -\mathrm{i}\overline{\nu_3} & -\gamma \end{bmatrix},$$

$\alpha, \beta, \gamma \in \mathbb{R}, \nu_1, \nu_2, \nu_3 \in \mathbb{C}$, in the first two rows and columns and let $J' = \mathrm{diag}(1, 1)$. Observe that $W_{J'}(A')$, which is a subset of $W_J(A)$, is a line segment with endpoints $\mathrm{i}\left(\frac{\alpha+\beta}{2} \pm \sqrt{\frac{(\alpha-\beta)^2}{4} + |\nu_1|^2}\right)$. If $\alpha = 1, \beta = 0, \nu_1 = 0$, then this line segment is $[0, \mathrm{i}]$, and

$$A = H^J + \mathrm{i}K^J = \begin{bmatrix} \mathrm{i} & 0 & \nu_2 \\ 0 & 0 & \nu_3 \\ \overline{\nu_2} & \overline{\nu_3} & -a' - \mathrm{i}\gamma \end{bmatrix},$$

where $-a' < 0$. Without loss of generality, we may assume that $c_1 = \nu_2 > 0, c_2 = \nu_3 > 0$. Hence, A is of the asserted form. □

If $\partial W_J(A)$ has a flat portion constituted by two half-lines of the same line, then one of the half-lines must be contained in $\partial W_J^+(A)$ and the other one in $\partial W_{-J}^+(A)$. This is an obvious consequence of the convexity of $W_J^+(A)$ and $W_{-J}^+(A)$.

Theorem 3. *Let $J = \mathrm{diag}(1, 1, -1)$ and let $A \in M_3$ be J-unitarily irreducible. Under J-unitary similarity, translation, rotation, and scaling, A may be written in the form*

$$A = \begin{bmatrix} a + \mathrm{i}\alpha & b & c \\ -b & \mathrm{i} & 0 \\ c & 0 & 0 \end{bmatrix}, \tag{6}$$

where $\alpha \in \mathbb{R}$ and $a, b, c > 0$, if and only if $W_J(A)$ has two closed half-lines of the same line on its boundary. In this form, $W_J^+(A)$ is contained in the closed right half-plane, the half-line of the positive imaginary axis with endpoint i *is contained in $\partial W_J^+(A)$, while the closed negative imaginary axis belongs to $\partial W_{-J}^+(A)$.*

Proof. ($\Rightarrow$) Let A be of the asserted form. Then

$$JH^J = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad JK^J = \begin{bmatrix} \alpha & -\mathrm{i}b & -\mathrm{i}c \\ \mathrm{i}b & 1 & 0 \\ \mathrm{i}c & 0 & 0 \end{bmatrix}.$$

Since $a > 0$, we have $W_J^+(H^J) = [0, +\infty[$, $W_{-J}^+(H^J) =]-\infty, 0]$. On the other hand, $\xi^* JH^J\xi$ vanishes if $\xi = (0, \zeta, \eta) \in \mathbb{C}^3$. For ξ of the above form, we obtain

$$\frac{\xi^* JK^J\xi}{\xi^* J\xi} = \frac{|\zeta|^2}{|\zeta|^2 - |\eta|^2}.$$

If $\xi^* J\xi < 0$ this quotient describes $]-\infty, 0]$, while if $\xi^* J\xi > 0$ it describes the interval $[1, +\infty[$. Thus, $W_J^+(A)$ is contained in the closed right half-plane and the asserted half-line is contained in this set. On the other hand, $W_{-J}^+(A)$ is contained in the closed left half-plane and the negative imaginary axis belongs to this set.

($\Leftarrow$) Without loss of generality, we may assume that $W_J(A)$ has the asserted closed half-lines on its boundary. Let $\{e_1, e_2, e_3\}$ be a J-orthogonal basis of $\mathbb{C}^3$ satisfying $H^J e_2 = 0$, $e_1^* Je_1 = e_2^* Je_2 = 1$, $e_3^* Je_3 = -1$. Consider the matrix representation of JH^J in this basis

$$JH^J = \begin{bmatrix} a & 0 & c \\ 0 & 0 & 0 \\ \bar{c} & 0 & b \end{bmatrix},$$

where a, b are real and c is a complex number obeying $ab = |c|^2$. By the same technique used in Theorem 2, we necessarily have $|a| \neq |c|$, and so the principal submatrix of H^J in the first and third rows and columns has the eigenvalues 0 and $a - b$, with two linearly independent anisotropic associated eigenvectors, and therefore, it can be diagonalized by a J-unitary similarity. Thus, in a proper basis

$$JH^J = \begin{bmatrix} a' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{7}$$

or

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a' \end{bmatrix} \tag{8}$$

with $a' = a - b$. It can be easily seen that the form (8) leads to a contradiction, because it is incompatible with the existence of two half-rays on the boundary of $W_J(A)$, and so we necessarily have (7). Thus, $W_J^+(H^J) = [0, +\infty[$ and $W_{-J}^+(H^J) =]-\infty, 0]$, being $a' > 0$ since $W_J^+(A)$ is contained in the closed right half-plane. Let

$$JK^J = \begin{bmatrix} \alpha & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ \mathrm{i}\overline{\nu_2} & \mathrm{i}\overline{\nu_3} & \gamma \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \nu_1, \nu_2, \nu_3 \in \mathbb{C}.$$

Now, let $J' = \mathrm{diag}(1, -1)$ and consider the 2×2 principal submatrix of $A = H^J + \mathrm{i}K^J$

$$A' = \mathrm{i}\begin{bmatrix} \beta & -\mathrm{i}\nu_3 \\ -\mathrm{i}\overline{\nu_3} & -\gamma \end{bmatrix}.$$

By the hyperbolical range theorem, $W_{J'}(A')$ reduces to two half-rays on the imaginary axis with endpoints $\mathrm{i}\left(\frac{\beta-\gamma}{2} \pm \sqrt{\frac{(\beta+\gamma)^2}{4} - |\nu_3|^2}\right)$. These endpoints coincide with 0 and i when we choose a basis such that $\beta = 1$, $\gamma = 0$, $\nu_3 = 0$. □

Now we investigate the existence of a whole line in $\partial W_J^+(A)$, and derive a canonical form for A.

Theorem 4. *Let $J = \mathrm{diag}(1, 1, -1)$ and let $A \in M_3$ be J-unitarily irreducible. Under J-unitary similarity, translation, and rotation, A may be written in the form*

$$A = \begin{bmatrix} 0 & \nu_1 & \nu_2 \\ -\overline{\nu_1} & a' + \mathrm{i}\beta & \nu_3 \\ \overline{\nu_2} & \overline{\nu_3} & 0 \end{bmatrix}, \tag{9}$$

where $\nu_1, \nu_3 \in \mathbb{C}$, $\nu_2 \in \mathbb{C}\backslash\{0\}$, $\beta \in \mathbb{R}$, $a' > 0$, or in the form

$$A = \begin{bmatrix} \mathrm{i}\alpha & \nu_1 & \nu_2 \\ -\overline{\nu_1} & a + \mathrm{i}\beta & -a + \nu_3 \\ \overline{\nu_2} & a + \overline{\nu_3} & -a - \mathrm{i}\gamma \end{bmatrix}, \tag{10}$$

where $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$, $\alpha, \beta, \gamma \in \mathbb{R}$, $a > 0$, $\beta + \gamma + 2\,\mathrm{Im}\,\nu_3 = 0$, $\nu_1 + \nu_2 \neq 0$, if and only if $\partial W_J^+(A)$ coincides with a line. In these forms, $W_J^+(A)$ is contained in the closed right half-plane, being the imaginary axis the boundary of $W_J^+(A)$.

Proof. ($\Rightarrow$) According to the hypothesis, for A in the form (9) we have

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a' & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad JK^J = \begin{bmatrix} 0 & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ \mathrm{i}\overline{\nu_2} & \mathrm{i}\overline{\nu_3} & 0 \end{bmatrix}.$$

Since $a' > 0$, we have $W_J^+(H^J) = [0, +\infty[$ and $W_{-J}^+(H^J) =]-\infty, 0]$. Moreover, $\xi^* JH^J\xi = 0$ when $\xi = (\zeta, 0, \eta) \in \mathbb{C}^3$ and the quotient

$$\frac{\xi^* JK^J\xi}{\xi^* J\xi} = \frac{2|\nu_2||\zeta||\eta| \sin\theta}{|\zeta|^2 - |\eta|^2},$$

$\theta = \arg\nu_2 - \arg\zeta + \arg\eta$, describes the real line when ζ, η range over $\mathbb{C}$ since by hypothesis $\nu_2 \neq 0$.

For A in the form (10), we have

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & -a & a \end{bmatrix} \quad \text{and} \quad JK^J = \begin{bmatrix} \alpha & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ \mathrm{i}\overline{\nu_2} & \mathrm{i}\overline{\nu_3} & \gamma \end{bmatrix}.$$

Since $a > 0$, then $W_J^+(H^J) = [0, +\infty[$ and $W_{-J}^+(H^J) =]-\infty, 0[$. Moreover, $\xi^* JH^J\xi = 0$ if $\xi = (1, \eta, \eta) \in \mathbb{C}^3$, and so

$$\frac{\xi^* JK^J\xi}{\xi^* J\xi} = \alpha + (\beta + \gamma + 2\,\mathrm{Im}\,\nu_3)|\eta|^2 + 2|\nu_1 + \nu_2||\eta| \sin\phi,$$

$\phi = \arg(\nu_1 + \nu_2) + \arg \eta$, describes the real line when $\eta \in \mathbb{C}$, since by hypothesis the coefficient of $|\eta|^2$ is zero and $|\nu_1 + \nu_2| \neq 0$.

($\Leftarrow$) Suppose that $\partial W_J^+(A)$ coincides with the imaginary axis. Let $e_1 \in \mathbb{C}^3$ such that $H^J e_1 = 0$, $e_1^* J e_1 = 1$. Consider the matrix representation of JH^J in the J-orthogonal basis $\{e_1, e_2, e_3\}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & c \\ 0 & \bar{c} & b \end{bmatrix},$$

where a, b are real and c is a complex number satisfying $ab = |c|^2$. If we have $|a| \neq |c|$, then in a proper basis JH^J may be taken either in the form

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a' \end{bmatrix}$$

or in the form

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a' & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $a' = a - b$. The first case leads to a contradiction, because it gives rise to a line segment on the boundary. In the second case, we have, for $a' > 0$, $W_J^+(H^J) = [0, +\infty[$, $W_{-J}^+(H^J) =]-\infty, 0]$, and $\xi^* JH^J \xi = 0$ if $\xi = (\zeta, 0, \eta) \in \mathbb{C}^3$. Let

$$JK^J = \begin{bmatrix} \alpha & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ \mathrm{i}\overline{\nu_2} & \mathrm{i}\overline{\nu_3} & \gamma \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \nu_1, \nu_2, \nu_3 \in \mathbb{C}$$

and consider the principal submatrix of $A = H^J + \mathrm{i}K^J$

$$A' = \begin{bmatrix} \mathrm{i}\alpha & \nu_2 \\ \overline{\nu_2} & -\mathrm{i}\gamma \end{bmatrix}.$$

For $J' = \mathrm{diag}(1, -1)$, then $W_{J'}(A')$ is the imaginary axis if $(\alpha + \gamma)^2 - 4|\nu_2|^2 < 0$, and without loss of generality we may take $\alpha = \gamma = 0$, $\nu_2 \neq 0$, and so

$$A = \begin{bmatrix} 0 & \nu_1 & \nu_2 \\ -\overline{\nu_1} & a' + \mathrm{i}\beta & \nu_3 \\ \overline{\nu_2} & \overline{\nu_3} & 0 \end{bmatrix}.$$

If $|a| = |c|$, then JH^J may be taken in the form

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & -a & a \end{bmatrix}.$$

For $a > 0$, we get $W_J^+(H^J) = [0, +\infty[$, $W_{-J}^+(H^J) =]-\infty, 0[$. On the other hand, if $\xi = (1, \eta, \eta) \in \mathbb{C}^3$ then $\xi^* JH^J \xi = 0$. Let

$$JK^J = \begin{bmatrix} \alpha & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ \mathrm{i}\overline{\nu_2} & \mathrm{i}\overline{\nu_3} & \gamma \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \nu_1, \nu_2, \nu_3 \in \mathbb{C}$$

and

$$f(\xi) := \frac{\xi^* J K^J \xi}{\xi^* J \xi} = \alpha + (\beta + \gamma + 2\,\mathrm{Im}\,\nu_3)|\eta|^2 + 2|\eta||\nu_1 + \nu_2|\sin\phi,$$

where $\phi = \arg\eta + \arg(\nu_1 + \nu_2) \in \mathbb{R}$. This function describes the imaginary axis if $\beta + \gamma + 2\,\mathrm{Im}\,\nu_3 = 0$ and $\nu_1 + \nu_2 \neq 0$. Hence, A has the asserted form. $\square$

We note that if A is of the form (9), then the imaginary axis is also a flat portion on $\partial W^+_{-J}(A)$. However, this is not true when A is of the form (10).

Now we investigate the existence of a single half-line on $\partial W^+_J(A)$ contained in the closed right half-plane, and derive a canonical form for A.

Theorem 5. *Let $J = \mathrm{diag}(1, 1, -1)$ and let $A \in M_3$ be J-unitarily irreducible. Under J-unitary similarity, translation, and rotation, A may be written in the form*

$$A = \begin{bmatrix} \mathrm{i}\alpha & \nu_1 & \nu_2 \\ -\overline{\nu_1} & a + \mathrm{i}\beta & -a + \nu_3 \\ \overline{\nu_2} & a + \overline{\nu_3} & -a - \mathrm{i}\gamma \end{bmatrix}, \tag{11}$$

where $\nu_1, \nu_2, \nu_3 \in \mathbb{C}$, $\alpha, \beta, \gamma \in \mathbb{R}$, $a > 0$, $\beta + \gamma + 2\,\mathrm{Im}\,\nu_3 > 0$, and

$$\alpha = \frac{|\nu_1 + \nu_2|^2}{\beta + \gamma + 2\,\mathrm{Im}\,\nu_3},$$

if and only if $W_J(A)$ has one closed half-line on its boundary. In this form, $W^+_J(A)$ has the positive imaginary axis as a flat portion and is contained in the closed right half-plane.

Proof. ($\Rightarrow$) According to the hypothesis

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & -a & a \end{bmatrix} \quad \text{and} \quad JK^J = \begin{bmatrix} \alpha & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ \mathrm{i}\overline{\nu_2} & \mathrm{i}\overline{\nu_3} & \gamma \end{bmatrix}.$$

Since $a > 0$, it follows that $W^+_J(H^J) = [0, +\infty[$, $W^+_{-J}(H^J) =]-\infty, 0[$. We have $\xi^* J H^J \xi = 0$ for $\xi = (1, \eta, \eta) \in \mathbb{C}^3$, and we easily obtain

$$f(\xi) := \frac{\xi^* J K^J \xi}{\xi^* J \xi} = \alpha + (\beta + \gamma + 2\,\mathrm{Im}\,\nu_3)|\eta|^2 + 2|\eta||\nu_1 + \nu_2|\sin\phi,$$

where $\phi = \arg\eta + \arg(\nu_1 + \nu_2)$. This function ranges over the positive imaginary axis because $\beta + \gamma + 2\,\mathrm{Im}\,\nu_3$ is positive and $\alpha = |\nu_1 + \nu_2|^2/(\beta + \gamma + 2\,\mathrm{Im}\,\nu_3)$.

($\Leftarrow$) Let the positive imaginary axis be a flat portion on $\partial W^+_J(A)$. Let $e_1 \in \mathbb{C}^3$ be such that $H^J e_1 = 0$, $e_1^* J e_1 = 1$. Consider the matrix representation of JH^J in the J-orthogonal basis $\{e_1, e_2, e_3\}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & c \\ 0 & \bar{c} & b \end{bmatrix},$$

where a, b are real and c is a complex number satisfying $ab = |c|^2$. We cannot have $|a| \neq |c|$, because under this assumption we are lead to the cases treated in Theorems 2,3,4. Thus, $|a| = |c|$ and in a proper basis

$$JH^J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & -a & a \end{bmatrix}.$$

For $a > 0$, we get $W_J^+(H^J) = [0, +\infty[$, $W_{-J}^+(H^J) =]-\infty, 0[$. Let

$$JK^J = \begin{bmatrix} \alpha & -\mathrm{i}\nu_1 & -\mathrm{i}\nu_2 \\ \mathrm{i}\overline{\nu_1} & \beta & -\mathrm{i}\nu_3 \\ \mathrm{i}\overline{\nu_2} & \mathrm{i}\overline{\nu_3} & \gamma \end{bmatrix}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \quad \nu_1, \nu_2, \nu_3 \in \mathbb{C}.$$

We easily find that $\xi^* JH^J \xi = 0$ for $\xi = (1, \eta, \eta) \in \mathbb{C}^3$, and we obtain

$$f(\xi) := \frac{\xi^* JK^J \xi}{\xi^* J \xi} = \alpha + (\beta + \gamma + 2\,\mathrm{Im}\,\nu_3)|\eta|^2 + 2|\eta||\nu_1 + \nu_2| \sin\phi$$

with $\phi = \arg\eta + \arg(\nu_1 + \nu_2) \in \mathbb{R}$. If $\beta + \gamma + 2\,\mathrm{Im}\,\nu_3 > 0$, then $f(\xi)$ describes a half-line of the form $[b', +\infty[$. Taking $\alpha = |\nu_1 + \nu_2|^2/(\beta + \gamma + 2\,\mathrm{Im}\,\nu_3)$, we have $b' = 0$. $\square$

3. $W_J(A)$ for J-unitarily reducible 3×3 triangular matrices

We denote by $\mathrm{Tr}\,\mathscr{C}_2(B)$ the sum of the 2×2 principal minors of a matrix B. Easy calculations show that:

Lemma 1. *For $A = H^J + \mathrm{i}K^J \in M_3$ and $J = I_r \oplus -I_{3-r}$ $(0 \leqslant r \leqslant 3)$*

$$\begin{aligned} F_A^J(u, v, w) = {} & w^3 + \det(H^J)u^3 + \det(K^J)v^3 + \mathrm{Re}\,\mathrm{Tr}(A)uw^2 + \mathrm{Im}\,\mathrm{Tr}(A)vw^2 \\ & + \mathrm{Im}\,\mathrm{Tr}\,\mathscr{C}_2(A)uvw + \mathrm{Tr}\,\mathscr{C}_2(H^J)u^2w + \mathrm{Tr}\,\mathscr{C}_2(K^J)v^2w \\ & + [\det(H^J) - \mathrm{Re}\det(A)]uv^2 + [\det(K^J) + \mathrm{Im}\det(A)]u^2v. \end{aligned}$$

If $A \in M_3$ is J-unitarily reducible, then there exists a matrix $U \in \mathscr{U}_{2,1}$ such that $U^{-1}AU = A_1 \oplus A_2$, and either the diagonal block A_1 has size 2 – Case 1, or size 1 – Case 2. First we analyze Case 1.

Theorem 6. *Let $J = \mathrm{diag}(1, 1, -1)$ and let*

$$A = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \in M_3.$$

The associated curve $C_J(A)$ is the union of the ellipse E (possibly degenerating into a disk) with foci a, b, minor axis of length s, and the point c if and only if

(1) $s^2 = |d|^2 - |e|^2 - |f|^2 > 0$ *and*
(2) $s^2 c = c|d|^2 - b|e|^2 - a|f|^2 + d\bar{e}f$.

Proof. Consider the matrix

$$B = \begin{bmatrix} a & s & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}, \quad s > 0,$$

whose associated curve $C_J(B)$ is the union of the ellipse E with foci a, b, minor axis of length s, and the point c.

Using Lemma 1, we conclude that the polynomials $F_A^J(u, v, w)$ and $F_B^J(u, v, w)$ have the same coefficients, except possibly the coefficients of u^3, v^3, u^2w and v^2w. Moreover, the coefficients of u^2w and v^2w in both polynomials are equal if and only if

$$s^2 = |d|^2 - |e|^2 - |f|^2 > 0.$$

On the other hand, the corresponding coefficients of u^3, v^3 are equal if and only if

$$s^2c = c|d|^2 - b|e|^2 - a|f|^2 + d\bar{e}f.$$

Hence, conditions (1) and (2) are necessary and sufficient for the matrices A and B to have the same associated curves. $\square$

Remark 1. To obtain an invariant form of conditions (1) and (2) in Theorem 6, note that

$$|d|^2 - |e|^2 - |f|^2 = \mathrm{Tr}(JA^*JA) - (|a|^2 + |b|^2 + |c|^2); \tag{12}$$

$$\begin{aligned} c|d|^2 - b|e|^2 - a|f|^2 + d\bar{e}f = {} & (|d|^2 - |e|^2 - |f|^2)\mathrm{Tr}\,A - \mathrm{Tr}(JA^*JA^2) \\ & + (a|a|^2 + b|b|^2 + c|c|^2). \end{aligned} \tag{13}$$

Thus, the following reformulation holds for conditions (1) and (2) and the theorem holds for matrices in M_3 that are J-unitarily triangularizable:

(1′) $s^2 = \mathrm{Tr}(JA^*JA) - (|a|^2 + |b|^2 + |c|^2)$ and
(2′) $s^2c = s^2\,\mathrm{Tr}\,A - \mathrm{Tr}(JA^*JA^2) + (a|a|^2 + b|b|^2 + c|c|^2)$.

Denote by $\sigma_J^+(A)$ ($\sigma_J^-(A)$) the set of eigenvalues of $A \in M_n$ with associated eigenvectors with positive (negative) J-norms.

Corollary 1. *Under the assumptions of Theorem 6, $W_J(A)$ is a "cone-like" figure (the pseudo-convex hull of E and c) if and only if c lies outside E; and it is the whole complex plane if and only if c lies inside E.*

Proof. Conditions (1) and (2) are equivalent to $C_J(A)$ being the union of the ellipse E and the point c. $W_J(A)$ is the pseudo-convex hull of c and E. If c is inside E, then $W_J(A)$ is the complex plane, because $c \in \sigma_J^-(A)$ and the ellipse is generated by vectors with positive J-norms. If c lies outside E, then $W_J(A)$ is a "cone-like" figure. $\square$

We observe that under the assumptions on J and A, $W_J(A)$ may be neither an elliptical disk nor a circular disk. Now we investigate when $C_J(A)$ consists of a hyperbola and a point (Case 2).

Theorem 7. *Let* $J = \mathrm{diag}(1, 1, -1)$ *and let*

$$A = \begin{bmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix} \in M_3.$$

The associated curve $C_J(A)$ consists of the point a and the hyperbola with foci b, c and non-transverse axis of length s if and only if

(1) $s^2 = -|d|^2 + |e|^2 + |f|^2 > 0$ *and*
(2) $s^2 a = -c|d|^2 + b|e|^2 + a|f|^2 - d\bar{e}f$.

Proof. Consider the matrix

$$B = \begin{bmatrix} a & 0 & 0 \\ 0 & b & s \\ 0 & 0 & c \end{bmatrix} \in M_3, \quad s > 0,$$

whose associated curve is the point a and the hyperbola with foci b and c and non-transverse axis of length s. The proof follows analogous steps to the proof of Theorem 6. □

Remark 2. Recalling (12) and (13), we obtain an invariant form of conditions (1) and (2) in Theorem 7:

(1′) $s^2 = -\operatorname{Tr}(JA^*JA) + |a|^2 + |b|^2 + |c|^2$ and
(2′) $s^2 a = -s^2 \operatorname{Tr} A + \operatorname{Tr}(JA^*JA^2) - (a|a|^2 + b|b|^2 + c|c|^2)$.

Corollary 2. *Under the assumptions of Theorem 7, denote by H_1 (H_2) the branch of H containing b (c) inside. Then $W_J(A)$ is:*

(1) $\mathbb{C}$ *if and only if a is inside H_2;*
(2) *the hyperbolical region limited by H if and only if a is inside H_1;*
(3) *a "cone-like" figure (the pseudo-convex hull of H and a) if and only if a is outside H.*

Proof. Under the hypothesis, conditions (1) and (2) in Theorem 7 are equivalent to $C_J(A)$ being the union of the hyperbola H and the point a. Since $W_J(A)$ is the pseudo-convex hull of a and H, and recalling that the point $a \in \sigma_J^+(A)$, we conclude that $W_J(A)$ coincides with the complex plane if the point a lies inside H_2; if a lies inside H_1, then the pseudo-convex hull of a and H is the hyperbolical region limited by H; finally, if a lies outside H, then $W_J(A)$ is a "cone-like" figure. □

The case of a triangular matrix with a triple eigenvalue is particularly simple.

Proposition 3. *Let* $J = \operatorname{diag}(1, 1, -1)$ *and*

$$A = \begin{bmatrix} p & q & r \\ 0 & p & s \\ 0 & 0 & p \end{bmatrix} \in M_3.$$

If at least one of the entries q, r or s is nonzero, then $W_J(A)$ coincides with $\mathbb{C}$. Otherwise, the set reduces to $\{p\}$.

Proof. Obviously, if $q = r = s = 0$, then $W_J(A) = \{p\}$. If $s \neq 0$, let $A' = A[2, 3]$ and $J' = \operatorname{diag}(1, -1)$. Then $W_{J'}(A') \subseteq W_J(A)$ and by the hyperbolical range theorem $W_{J'}(A')$ is the complex plane. The case $r \neq 0$, may be analogously treated considering $A' = A[1, 3]$ and $J' = \operatorname{diag}(1, -1)$. If $q \neq 0$, we take $A' = A[1, 2]$ and $J' = \operatorname{diag}(1, 1)$. By the elliptical range theorem, $W_{J'}(A')$ is a disc centered at p with radius $|q|/2$. The point $p \in \sigma_J^-(A)$ is in the interior of the disc, and since the disc is generated by vectors with positive J-norm, the pseudo-convex hull of the disc and of the point p is the whole complex plane. □

4. Examples

We present illustrative examples of the obtained results. The figures were produce with *Mathematica* 5.1, and the boundaries of the convex sets $W_J^+(A)$ and $W_{-J}^+(A)$ are represented by thick lines.

Example 1. Let

$$A = \begin{bmatrix} \mathrm{i} & 0 & 1/2 \\ 0 & 0 & 1/2 \\ 1/2 & 1/2 & -\sqrt{2} \end{bmatrix}.$$

Easy calculations show that

$$F_A^J(u, v, w) = v^3/4 + (v - 2\sqrt{2}u)vw/2 + (v - \sqrt{2}u)w^2 + w^3.$$

The associated curve $C_J(A)$, represented in Fig. 1, is quartic with a real cusp, being the imaginary axis a double tangent. The set $W_J^+(A)$ is contained in the closed right half-plane and it is the convex hull of the branch of $C_J(A)$ in this half-plane. The line segment $[0, i]$ is a flat portion on $\partial W_J^+(A)$. On the other hand, $W_{-J}^+(A)$ is contained in the half-plane $\{z \in \mathbb{C} : \mathrm{Re}\, z \leqslant -\sqrt{2}\}$, being the convex hull of the branch of $C_J(A)$ in that region (see Theorem 2).

Example 2. Consider, now, the matrix

$$A = \begin{bmatrix} 2 & 1 & 1/2 \\ -1 & \mathrm{i} & 0 \\ 1/2 & 0 & 0 \end{bmatrix}$$

with $F_A^J(u, v, w) = v^3/4 - 3v^2w/4 + (vw + w^2)(2u + w)$. The associated curve $C_J(A)$, represented in Fig. 2, is quartic with a real cusp and the imaginary axis is a double tangent of the curve. Its pseudo-convex hull originates half-lines on $\partial W_J^+(A)$ and on $\partial W_{-J}^+(A)$, being $W_J^+(A)$ ($W_{-J}^+(A)$) contained in the closed right half-plane (closed left half-plane) (see Theorem 3).

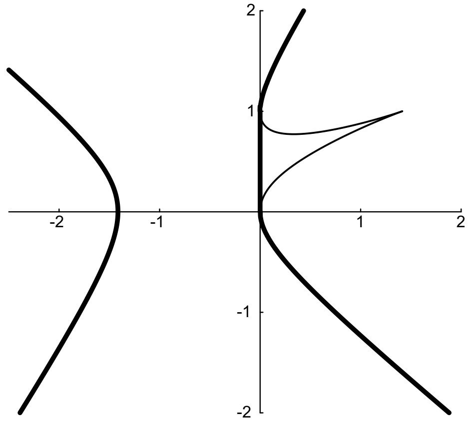

Fig. 1. The line segment $[0, i]$ is a flat portion on $\partial W_J^+(A)$.

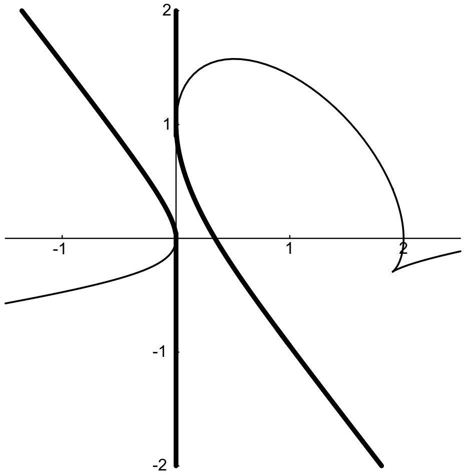

Fig. 2. The negative imaginary axis is a flat portion on $\partial W^+_{-J}(A)$ and the half-line of the positive imaginary axis with endpoint i is a flat portion on $\partial W^+_J(A)$.

Example 3. Let

$$A = \begin{bmatrix} 0 & 1 & 1/2 \\ -1 & 1 & 0 \\ 1/2 & 0 & 0 \end{bmatrix},$$

where $F^J_A(u, v, w) = -3v^2w/4 + u(v^2/4 + w^2) + w^3$. The associated curve $C_J(A)$, represented in Fig. 3, is quartic with three real cusps and the imaginary axis is a double tangent of the curve (at complex points). This example leads to a degenerate case, since $W^+_{-J}(A) = \{z \in \mathbb{C}: \operatorname{Re} z \leqslant 0\}$ and $W^+_J(A) = \{z \in \mathbb{C}: \operatorname{Re} z \geqslant 0\}$. The imaginary axis is a flat portion on $\partial W^+_J(A)$ and on $\partial W^+_{-J}(A)$ (see Theorem 4 (9)).

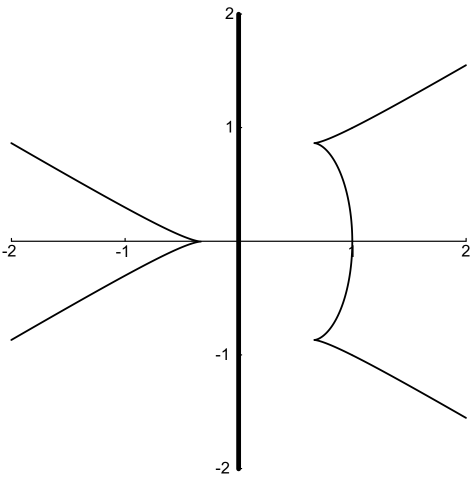

Fig. 3. The imaginary axis is a flat portion on $\partial W^+_J(A)$ and on $\partial W^+_{-J}(A)$.

Example 4. Let

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix},$$

where $F_A^J(u, v, w) = 4uv^2 + w^3$. The associated curve $C_J(A)$, illustrated in Fig. 4, is cubic with a real cusp and a real flex, both in the line of infinity. The flexional tangent is the imaginary axis. This example leads also to a degenerate case, because $W_{-J}^+(A) = \{z \in \mathbb{C}: \operatorname{Re} z < 0\}$ and $W_J^+(A) = \{z \in \mathbb{C}: \operatorname{Re} z \geqslant 0\}$. The imaginary axis is a flat portion on $\partial W_J^+(A)$ (see Theorem 4 (10)).

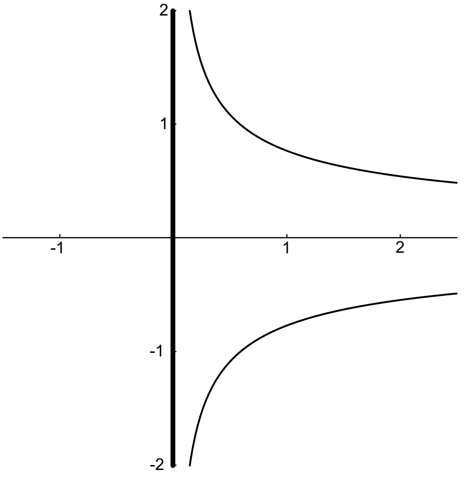

Fig. 4. The imaginary axis is a flat portion on $\partial W_J^+(A)$.

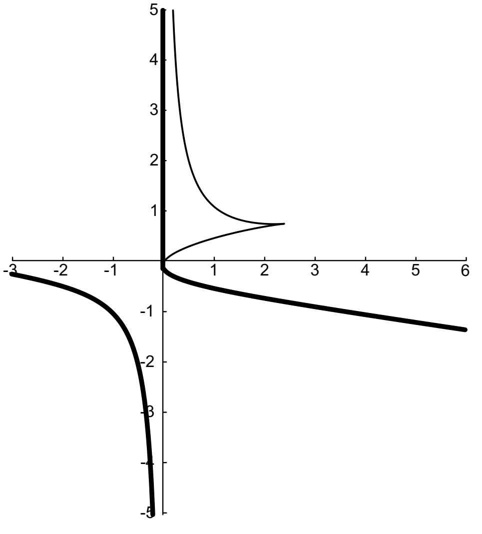

Fig. 5. The positive imaginary axis is a flat portion on $\partial W_J^+(A)$.

Example 5. Finally, consider the matrix

$$A = \begin{bmatrix} \mathrm{i}/16 & -1/2 & 0 \\ 1/2 & 1+\mathrm{i} & -1+\mathrm{i} \\ 0 & 1-\mathrm{i} & -1-\mathrm{i} \end{bmatrix}.$$

We get $F_A^J(u, v, w) = 16w^3 + vw^2 - 64uvw - 4v^2w + 4v^3$. The associated curve $C_J(A)$, represented in Fig. 5, is quartic with a real cusp, being the imaginary axis a double tangent (at the origin and at a point in the line of infinity). The set $W_J^+(A)$ ($W_{-J}^+(A)$) is contained in the closed right half-plane (open left half-plane), and it is the convex hull of the branch of $C_J(A)$ in this half-plane. The positive imaginary axis is a flat portion on $\partial W_J^+(A)$ (see Theorem 5).

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References

[1] N. Bebiano, R. Lemos, J. da Providência, G. Soares, On generalized numerical ranges of operators on an indefinite inner product space, Linear and Multilinear Algebra, 52 (2004) 203–233.
[2] N. Bebiano, R. Lemos, J. da Providência, G. Soares, On the geometry of numerical ranges in spaces with an indefinite inner product, Linear Algebra Appl. 399 (2005) 17–34.
[3] N. Bebiano, J. da Providência, R. Teixeira, Indefinite numerical range of 3 × 3 matrices, submitted for publication.
[4] G. Fischer, Plane Algebraic Curves, American Mathematical Society, Providence, 2001.
[5] R.A. Horn, C.R. Johnson, Topics in Matrix Analysis, Cambridge University Press, New York, 1991.
[6] D. Keeler, L. Rodman, I. Spitkovsky, The numerical range of 3 × 3 matrices, Linear Algebra Appl. 252 (1997) 115–139.
[7] R. Kippenhahn, Über den wertevorrat einer matrix, Math. Nachr. 6 (1951) 193–228.
[8] C.-K. Li, L. Rodman, Remarks on numerical ranges of operators in spaces with an indefinite metric, Proc. Amer. Math. Soc. 126 (1998) 973–982.
[9] C.-K. Li, L. Rodman, Shapes and computer generation of numerical ranges of Krein space operators, Electron. J. Linear Algebra 3 (1998) 31–47.
[10] C.-K. Li, N.K. Tsing, F. Uhlig, Numerical ranges of an operator in an indefinite inner product space, Electron. J. Linear Algebra 1 (1996) 1–17.
[11] F.D. Murnaghan, On the field of values of a square matrix, Proc. Natl. Acad. Sci. USA 18 (1932) 246–248.
[12] P. Psarrakos, Numerical range of linear pencils, Linear Algebra Appl. 317 (2000) 127–141.